Experimental and Theoretical Comparison of Photon-Counting and Current Measurements of Light Intensity

R. Jones, C. J. Oliver, and E. R. Pike

The effect of gain variation on the integrated output-charge distribution of a photomultiplier tube is investigated experimentally and shown to be a predictable function of the multiplier single-electron response. Standardized or nonstandardized pulses recorded using either capacitive or digital storage are considered. Theoretical values for the moment-generating functions and variances (noise powers) of the charge distributions obtained in these four cases are given, and the role of these various distributions in determining the length of time required to achieve a given accuracy in a light-flux measurement is discussed. The experimental measurements adequately confirm the theoretical predictions. The work includes a critical discussion of the field of theoretical and experimental noise investigations in photomultiplier tubes with regard to their relevance in the present state of technology.

Introduction

There are two important factors affecting the time taken to achieve a desired accuracy when using a photomultiplier to measure light flux. They are, first, whether the detector is to be used directly as a current source or whether the output is to be converted into a standardized-pulse output from a discriminator. Second, one needs to consider whether it is to be used with storage, having its own internal time constant governing integration time, (e.g., a microammeter, ratemeter, or pen-recorder), or to be used with accumulating storage in which no information is lost (e.g., a scalar or multichannel pulse-height analyzer). For the purposes of this paper these storage techniques will be referred to as *capacitive* and *digital*, respectively. The significance of the choice between these different techniques is discussed from both experimental and theoretical viewpoints. This work was stimulated by a recent paper by Rolfe and Moore,[1] who draw experimental conclusions that we find unacceptable with respect to the merits of the various techniques.

In the original papers of Zworykin *et al.*[2] (1936) and Shockley and Pierce[3] (1938) expressions were derived for the noise in the output charge q of an electron multiplier as a function of its secondary-emission properties. They considered both Poisson[2,3] secondary-emission statistics and generalized statistics giving a different mean-square deviation.[3] Their experimental evidence for one-, two-, and three-stage tubes showed good agreement with the theory. Unfortunately the development of this device into practical photomultiplier tubes introduced various problems, in particular different light- and dark-count pulse-height distributions $P(q)$ and correlations between dark counts. Much of the work in the literature devoted to the achievement of the optimum performance of photomultiplier tubes has concentrated on these two departures from ideality and has even ignored the effects of the secondary-emission statistics described by the original authors. Modern tubes, however, are available that have effectively the same distribution for light and dark counts, even when cooled, and have Poissonian dark-count statistics (eg, the ITT FW130[4,5]). These are mainly tubes that have small photocathode areas so that not only are the number of nonthermionic dark pulses, caused by the Cerenkov radiation of cosmic rays in the tube walls[6,7] and possibly disintegration of ^{40}K, very low (e.g., $\sim$0.5/sec for the ITT FW130), but the light piping effect is reduced also. This causes the pulses that are observed to correspond only to single photoelectrons,[4] not to the large bunches that have been observed by Young[7] with large photocathode tubes. (With an ITT FW118, having similar geometry but with an S-1 rather than an S-20 photocathode, Young[7] also observed no dark pulses larger than those corresponding to single photoelectrons.) Another factor is that for good tubes the low-pulse-height tails in the $P(q)$'s are not now significant and are, to the extent that they are observed, probably not due to the nonthermionic effects of positive ions but to some electron-optical effect.[8,9] Accordingly, the work of many authors on the problem of low-light detection and the

The authors are with the Royal Radar Establishment, Malvern, Worcestershire, England.

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Table I. Bibliography of Original Theoretical Work on the Factors Affecting the Integrated-Charge Fluctuations from Photomultiplier Tubes Detecting Coherent or Wideband Thermal Light[a,b]

Detection method	Digital storage	Capacitive storage	$P(q)$light ≠ $P(q)$dark	Dark-count statistics
Standardized (photon counting)	*S18*[c] *M58*	*Ta51*	B62[d]	Tl 68[d]
Non-standardized (dc)	*SP38*[e] *Present paper*	*Present paper*	B62[d]	Tl 68[d]
Shot-noise power	Y69	PG67[f]	PG67[d,f]	Y69[d]

[a] B62, Baum[24]; M58, Mandel[12]; PG67, Pao and Griffiths[13]; S18, Schottky[10]; SP38, Shockley and Pierce[3]; Ta51, Taylor[18]; T168, Tull[16]; Y69, Young.[11]

[b] The region *enclosed by a double line* is relevant to operation of an ideal photomultiplier.

[c] Equivalent calculation.

[d] Dark-count limited operation.

[e] Variance only.

[f] Ignores effect of gain distribution.

comparison of photon counting with other techniques is largely outdated for high-quality tubes, since the main remaining source of noise introduced is the statistical variation of gain in the multiplier.

When using a photomultiplier to measure light flux, the detector output, consisting of a train of pulses of varying heights corresponding to the detection of photons, is stored in some fashion for an integration time T. The over-all noise in signal is determined by the variance of this integrated charge. Additional noise results from the detector dark counts. Table I is a bibliography of original work, so far as we can establish, on the theoretical effect of various factors on the integrated-charge fluctuations. Three basic types of detection method are considered, namely, the use of standardized output pulses (photon-counting), non-standardized output pulses (dc techniques), and shot-noise power measurement. Comparison of these techniques for noise-in-signal limited operation shows that the signal-to-noise power ratios in the three cases are in the proportions 1 (Ref. 10): $1 + [\mathrm{Var}(q)/\bar{q}^2]$ (Ref. 3):$1 + [\mathrm{Var}(q^2)/\bar{q}^{2^2}]$ (Ref. 11), respectively, for either storage technique. This shows that the shot-noise power technique would be poorer than normal dc operation, which, in turn, would be poorer than photon counting under these circumstances. The results given in italics in the table are the only ones relevant to the best present-day tubes.

The effect of the photon-noise arising in detection of coherent or broadband thermal light was analyzed by Mandel,[12] though an equivalent expression for shot noise was derived long before by Schottky.[10] This simple theory was extended to allow for the multiplier gain distribution by Shockley and Pierce,[3] who derived an expression for the variance of the integrated-charge distribution. In the present paper, to be complete, we present in an Appendix a derivation of the moment-generating function of this distribution.

For a multiplier having different light- and dark-count charge distributions, the shot-noise power method could result in an improvement of the signal-to-noise ratio (SNR) (Pao and Griffiths[13]). Pao and Griffiths's analysis, however, was incomplete since it ignored the effect of the gain distribution and only considered the effect of different mean gains for light and dark pulses. If the tube has correlations between pulses it has been shown[11] that analog methods would be badly affected, the shot-noise method being worse than dc methods. Non-Poissonian dark-count statistics, such as have been observed by Rodman and Smith[14] and Gadsden,[15] were included by Tull[16] in his analysis of the dc detection scheme. However, to the extent that it is now possible to obtain detectors not suffering from these nonidealities, it is only really necessary to consider the relative merits of dc or photon-counting detection with noise-in-signal limited operation.

The difference between digital and capacitive storage is also an important consideration. There appears to be some confusion in the literature between inverse bandwidth and digital storage time; see, for example, Alfano and Ockman,[17] who derive a greater SNR for dc detection than for photon counting. The theory of the capacitive storage of standardized pulses, which is basically that of a ratemeter, was given by Taylor.[18] Again, to be complete, we give a derivation, including the effects of varying pulse height, in the Appendix.

Table II contains a bibliography of original work on the experimental measurement of the properties con-

Table II. Bibliography of Original Experimental Work on the Factors Affecting the Integrated-Charge Fluctuations from Photomultiplier Tubes Detecting Coherent or Wideband Thermal Light[a,b]

Detection method	Digital storage	Capacitive storage	$P(q)$light ≠ $P(q)$dark	Dark-count statistics
Standardized (photon counting)	*FH65*[c] *MMHP 65*[c] *JMP 65*	*AO68*[c,d] *Present paper*	G65[d]	RS63[c,d] OP68[d]
Non-standardized (dc)	*Present paper*	*ZMM36*[e] *ESG 52*[f] *FJOP69*	NS68[d]	
Shot-noise power		PG67[d]		

[a] AO68, Alfano and Ockman[17]; ESG52, Engstrom *et al.*[29]; FH65, Freed and Haus[19]; FJOP69, Foord *et al.*[5]; G65, Gadsden[15]; JMP65, Johnson *et al.*[21]; MMHP65, Marguin *et al.*[20]; NS68, Nakamura and Schwarz[22]; OP68, Oliver and Pike[4]; PG67, Pao and Griffiths[13]; RS63, Rodman and Smith[14]; ZMM36, Zworykin *et al.*[2].

[b] The region *enclosed by a double line* is relevant to operation of an ideal photomultiplier.

[c] Low statistical accuracy.

[d] Dark-count limited operation.

[e] 1–3 stage multipliers only.

[f] Includes the effect of collection efficiency.

sidered in Table I. The original photon-counting experiments which investigated the photon noise involved in the detection of coherent, or broadband thermal, light were made by Freed and Haus,[19] Marguin *et al.*,[20] and Johnson *et al.*[21] Many authors[1,15,22,23] have made measurements on the variation of the SNR when photon counting as the discriminator threshold is varied. The fact that this ratio was not trivially related to the light $P(q)$ was, of course, due to the differing light and dark $P(q)$'s. Only Papayan and Rozanov[23] interpreted their results quantitatively in terms of these distributions. When comparing the SNR's obtained with photon counting and dc detection, most authors[17,22,23] observed an advantage in photon counting. However, except in Ref. 23 quantitative interpretation is difficult. In the strong-signal case both Nakamura and Schwarz[22] and Rolfe and Moore[1] found no advantage in photon counting, probably because they failed to count many of the photomultiplier output pulses. Baum,[24] Tull,[16] and Young[11] have predicted the advantage of photon counting by integration of experimental pulse-height distributions and obtained results that tended to favor photon counting as the multiplier tended to ideality.

In this paper we shall consider only noise-in-signal limited operation since, as we have said, practically ideal multipliers can be obtained having nearly identical light- and dark-count pulse-height distributions with no correlations between pulses. Operation of such multipliers in the dark-count limited region is in principle the same as noise-in-signal limited operation.[4] In this paper measurements of single-electron response $P(q)$, photon-counting distributions $p(n,T)$, and integrated-charge distributions $P(Q,T)$ are interpreted in terms of the SNR considerations first formulated by Zworykin *et al.*[2] and Shockley and Pierce[3] and since expanded by Prescott.[25] In each case the tubes are operated under noise-in-signal limited conditions with dark-count rates rendered negligible by cooling. Both digital and capacitive storage techniques are considered. The value of the relative variance of the integrated-charge distribution corresponds to the noise in signal-to-signal power ratio.

Theory

Let us consider in detail the parts played by the photomultiplier and the storage system. The optical radiation field is detected at the photocathode by the emission of photoelectrons corresponding to the annihilation of photons. This is a Poisson process, and so, for broadband thermal light or for coherent radiation, these photons will have a random time distribution.[19–21] However, the multiplication process depends on a further statistical distribution of secondary-emission probabilities giving rise to variation in the pulse heights observed at the anode. If the output of the multiplier were to be standardized in a discriminator in such a way that none of the pulses arising from photoelectrons emitted by the photocathode was lost and none added, then the standardized output would contain only the original time distribution of the photoelectrons, and the effects of varying pulse heights would have been eliminated. The total integrated charge Q arriving from the photomultiplier over a time T is made up of individual pulses of varying or fixed charge q. The relative variance of Q is then given by (see Appendix)

$$\left|\frac{\mathrm{Var}(Q)}{\bar{Q}^2}\right|_{\text{standarized}} = \frac{1}{\bar{n}T} \tag{1}$$

and

$$\left|\frac{\mathrm{Var}(Q)}{\bar{Q}^2}\right|_{\text{nonstandardized}} = \frac{1}{\bar{n}T}\left[1 + \frac{\mathrm{Var}(q)}{\bar{q}^2}\right], \tag{2}$$

where $\bar{n}$ is the photoelectron count rate, q is the charge per pulse, and $\mathrm{Var}(x)$ denotes the variance of the probability distribution of x. The relative variance of Q, discussed here, is the reciprocal of the so-called signal-to-noise in signal power ratio under the given conditions of integration. From Eqs. (1) and (2) one can see that the use of standardized pulses should lead to a reduction in $\mathrm{Var}(Q)/\bar{Q}^2$ and hence in the experimental time needed to achieve a certain accuracy, by a factor $[1 + \mathrm{Var}(q)/\bar{q}^2]$. In addition one would expect an improvement due to the removal of any charge-leakage effects, the rejection of pulses arising from elsewhere in the dynode chain, and, to a first order, the removal of the effects of drifts in the system gain.

In this discussion two effects have not been considered so far which have the same influence on each result. First, the detection process at the photocathode has a certain quantum efficiency that determines the best achievable performance of the detector. A second limitation is that the multiplier may fail to multiply some of the incident photoelectrons giving a reduced over-all efficiency for the detector. This collection efficiency is a parameter that cannot be improved by subsequent electronics and, again, affects equally all techniques for using photomultipliers. The above formulas are simply modified to take account of these effects by using in them the observed count rate $\bar{n}$.

It is well known from the theory of the ratemeter[18] (see also Whitford[26]) that the relative variance of the charge in a capacitive store (of time constant τ) is given by

$$\left|\frac{\mathrm{Var}(Q)}{\bar{Q}^2}\right|_{\text{standardized}} = \frac{1}{2\bar{n}\tau}. \tag{3}$$

For nonstandardized pulses this becomes (see Appendix)

$$\left|\frac{\mathrm{Var}(Q)}{\bar{Q}^2}\right|_{\text{nonstandardized}} = \frac{1}{2\bar{n}\tau}\left[1 + \frac{\mathrm{Var}(q)}{\bar{q}^2}\right]. \tag{4}$$

Equations (3) and (4) show that the choice of the integration time constant τ determines the relative variance of Q. To increase the signal-to-noise in signal ratio the time constant must be increased. Obviously, however, if the signal were changing, a large time constant would introduce a distortion of the result because of the exponential residual. Similarly, if only a limited period were available for a measurement, the choice of too long a time constant would prevent the output reaching a steady state and so also give rise to distortion as before. If we select a time constant giving 2%

Table III. Relative Merits of the Different Techniques of Using Photomultipliers

Integrated Charge Distributions		
Photomultiplier output	Storage	$\mathrm{Var}(Q)/\bar{Q}^2$
Nonstandardized	Capacitive[a]	$(2/\bar{n}T)\{1 + [\mathrm{Var}(q)/\bar{q}^2]\}$
Standardized	Capacitive[a]	$2/\bar{n}T$
Nonstandardized	Digital	$(1/\bar{n}T)\{1 + [\mathrm{Var}(q)/\bar{q}^2]\}$
Standardized	Digital	$1/\bar{n}T$

[a] The capacitive store time constant is set to be $\tau = T/4$ for 2% distortion.

exponential residual, as is commonly done in the use of the rate meter, then $\tau = T/4$. Under these conditions Eqs. (3) and (4) become

$$\left|\frac{\mathrm{Var}(Q)}{\bar{Q}^2}\right|_{\text{standardized}} = \frac{2}{\bar{n}T} \tag{5}$$

and

$$\left|\frac{\mathrm{Var}(Q)}{Q^2}\right|_{\text{nonstandardized}} = \frac{2}{\bar{n}T}\left[1 + \frac{\mathrm{Var}(q)}{\bar{q}^2}\right]. \tag{6}$$

Therefore, since the relative variance is doubled (accepting arbitrarily this level of distortion), one would expect that capacitive storage should require an experiment duration T, twice that needed with digital storage to achieve the same accuracy. Reduction of the capacitive time constant would reduce the distortion due to the exponential residual, but as can be seen from Eq. (3) and (4) this is only achieved at the expense of a higher value of the relative variance of Q, ie, a poorer signal-to-noise in signal ratio. These results are summarized in Table III and derived in detail in the Appendix. If we define photon counting as the use of standardized pulses with digital storage and current measurement as the use of nonstandardized pulses with some form of current meter (capacitive storage) with the above time constant then current measurement will require an experimental time a factor of $2\{1 + [\mathrm{Var}(q)/\bar{q}^2]\}$ greater than photon counting to achieve the same accuracy.

Apparatus and Methods

To measure charge distributions from photomultipliers a charge-sensitive integrating amplifier was designed. Used as a charge integrator (digital storage) it was allowed to charge up for a fixed time determined by an external clock (Fig. 1), sampled (using the sampled voltage analysis mode of a multichannel analyzer), and then reset. Alternatively, it was used as a capacitive store by introducing an internal time constant. If used with a shorter (~30-μsec) internal time constant it acted as a charge-sensitive pulse amplifier, in which case a low input count rate was required to avoid pulse-pileup effects. The amplifier input was connected either directly to the photomultiplier anode or via a second, wideband, amplifier and discriminator. This second amplifier had an input impedance of 50 Ω, a gain of approximately 100, and almost distortion-free characteristics in photon-counting applications because of its fast risetime (2 nsec) and good overload characteristics.[5] The same final charge-sensitive amplifier, therefore, was used to make all the measurements. Since the standardized input with externally fixed sample time provided the well-known photon-counting distribution for a Poisson source,[19] this acted as a check on the linearity and distortion of the amplifier which was found to be better than 1%.

Two types of photomultiplier are considered here: the ITT FW130 which has proved near ideal in our experience, having Poisson multiplication statistics, and the EMI 6256, the tube used by Rolfe and Moore[1] to obtain results leading to conclusions with which we do not agree.

Experimental Results

Pulse-Height Distributions

Figure 2 shows a pulse-height distribution $P(q)$, obtained for an ITT FW130 with a cathode to dynode 1 voltage of 300 V and a dynode 1 to anode voltage of

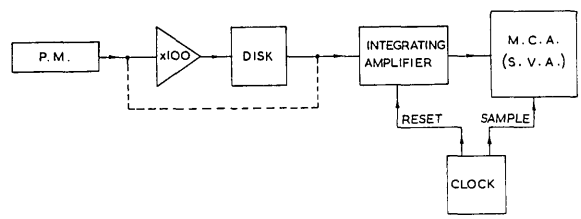

Fig. 1. Block diagram of the use of a charge-sensitive integrating amplifier to investigate the effects of pulse-height variation and type of storage on the accuracy of integrated charge measurements.

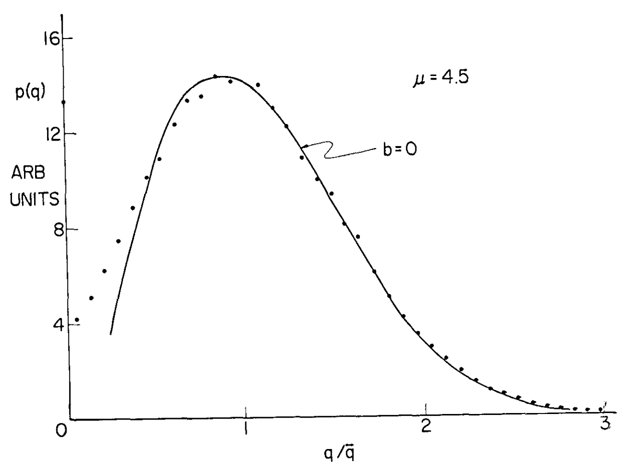

Fig. 2. Pulse-height distribution $P(q)$ for an ITT FW130. A theoretical distribution based on Prescott's analysis for $b = 0$ is also shown for comparison.

1800 V. In common with Barr and Eberhardt[27] this cathode to dynode 1 voltage was found to give the optimum pulse-height distribution. On standardizing the multiplier output the total number of pulses recorded dropped by approximately 2%, showing that nearly all pulses of the distribution of Fig. 2 were counted by the discriminator. The gain of the multiplier was measured by finding the discriminator setting that halved the observed count rate. The pulse shape was determined from a sampling oscilloscope trace, and hence the median charge per pulse could be calculated giving the multiplier gain with an accuracy that should be better than 10%. For the FW130 the median gain can be calculated from the pulse-height distribution of Fig. 2 to be about 2% less than the mean gain. This would give a negligible error in the calculated gain per stage. The variation of the first-stage gain with voltage, other stages being kept at constant gain, was found to be approximately proportional to $V^{0.7}$. Thus the gain of the first stage could be corrected for the nonuniform voltage distribution giving a value of 4.5. The relative variance of the observed pulse-height distribution $[\mathrm{Var}(q)/\bar{q}^2]$ was found to be 0.29 yielding a b parameter close to zero using Prescott's analysis.[25] The theoretically predicted $P(q)$ is also included in Fig. 2, and, considering that the tube is not operated with uniform gain per stage as the theory assumes, the agreement is satisfactory. The bulk of the residual pulses at low energy are thought to be due to elastic primaries.[9]

Figure 3 shows a pulse-height distribution for an EMI 6256 obtained with 1600 V divided according to the manufacturer's recommendation. The over-all gain was measured in the same way as for the FW130 yielding a value for the first-stage gain of 6.5. For this tube the relative variance was found to be 0.44, giving a value of 0.23 for the b parameter. Figure 3 also shows the theoretical curve, based on Prescott's analysis, for $b = 0.2$, the closest value computed, demonstrating reasonable agreement. The curve for $b = 0$ is also shown. Thus one would expect the 6256 to be noisier than the FW130 because of the increased variance of q. The fact that the distribution has a more pronounced low pulse-height tail than the FW130, used at the same mean gain, is reflected in the fact that our measurement of the standardized count rate from the discriminator showed that 5% of the distribution was lost, as opposed to only 2% with the FW130.

Our distribution for the 6256 is significantly more peaked than that obtained by Rolfe and Moore[1] with a similar tube voltage. One explanation could involve dependence on the area of photocathode illuminated since this affects the pulse-height distribution by increasing the possible dynode gain variation.[9,28] In this work we have illuminated a spot of 0.2-mm diam; illumination of a larger area gives a significantly poorer $P(q)$. The area used by Rolfe and Moore is not stated. However, this explanation is inconsistent with the results they obtained from the variation of counting rate with discriminator threshold, which indicates a peaked distribution. In addition, the mean pulse height for their two measurements is in disagreement since they found that the count rate was reduced to about 20% of its maximum value by adjusting the threshold of the discriminator to the *mean* pulse height, estimated from their pulse-height distribution. Even for an exponential SER one would expect the count rate to be reduced to only $\sim$35% of its maximum value under these conditions. For distributions with any type of peak a value nearer 50% is expected (as for the FW130). Rolfe and Moore offered no comment on these discrepancies.

Integrated Charge Distribution with Digital Storage —the Effect of Pulse-Height Variation

The effect of variation in pulse height on the integrated charge distribution was measured using an external clock (to control the sampling period) and sampled voltage analysis as described previously. This is essentially a digital storage technique corresponding to lines 3 and 4 of Table III. Typical integrated charge distributions for the FW130 and 6256 are in Figs. 4 and 5, respectively. They were built up from 10^4 to 10^5 samples, and in each case would be equivalent to taking the same number of well separated samples from a chart recording with a time constant half the integration time used here. The distribution for the FW130 shows a comparatively small difference between standardized and nonstandardized inputs—as expected for this clearly peaked distribution. From these results, using the corrected standardized count rate, the factors $[1 + \mathrm{Var}(q)/\bar{q}^2]$ can be calculated.

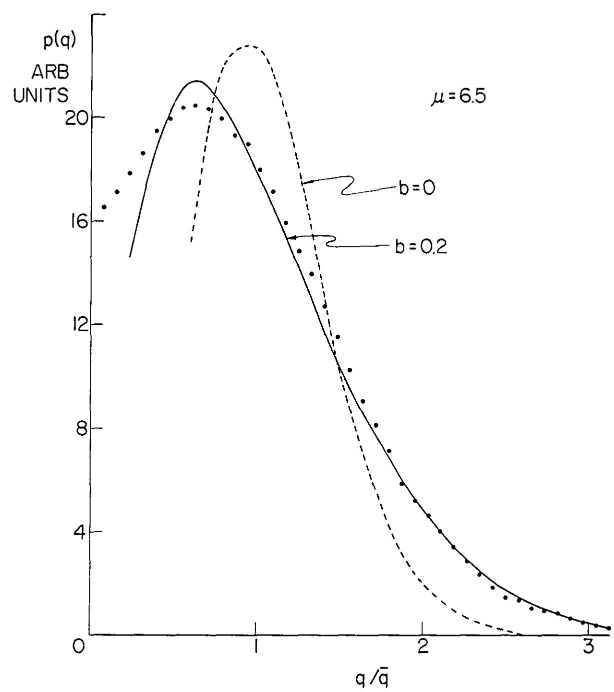

Fig. 3. Pulse-height distribution $P(q)$ for an EMI 6256. Theoretical distributions for $b = 0$ and 0.2 are also shown for comparison.

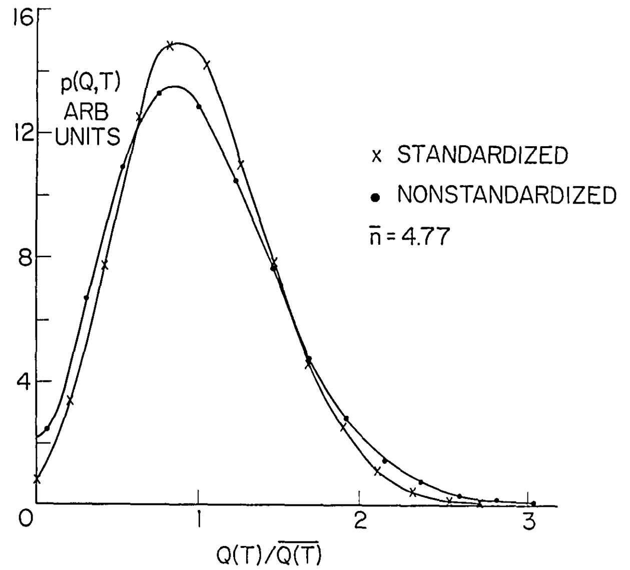

Fig. 4. Typical integrated-charge distributions $P(Q,T)$ obtained using digital storage (a multichannel analyzer) for both standardized and nonstandardized outputs from the FW130.

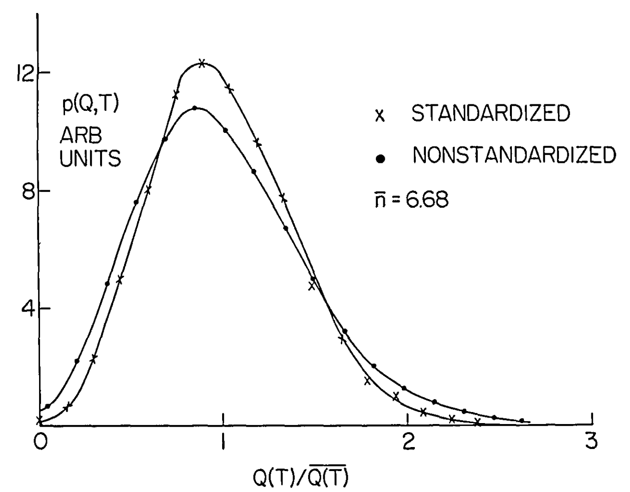

Fig. 5. Typical integrated-charge distributions $P(Q,T)$ for the 6256 for both standardized and nonstandardized outputs.

This experiment was repeated for a number of different counts per sample time for each tube, and the results are summarized graphically in Fig. 6. The ordinate gives the factor $1 + [\mathrm{Var}(q)/\bar{q}^2.]$ However, since the gains are not the same for the two photomultipliers, the expected value of this factor would differ, even if both tubes had a $P(q)$ corresponding to Poisson multiplication statistics ($b = 0$). In Fig. 6, therefore different scales have been used so that the ideal case is at the same reference level for each photomultiplier tube. In each case the standardized photon-counting distribution gives a factor of 1 as expected; departure from this value would indicate correlations between the output pulses. For the nonstandardized distributions the factors were found to be 1.30 (FW130) and 1.44 (6256). These increased variances show the increase in experimental time needed to attain the same statistical accuracy as could be attained by the use of standardized pulses. The fact that some pulses are lost in standardization reduces this advantage slightly.

If the pulse-height distributions obtained by Rolfe and Moore for the 6256 were correct one would expect their SNR for nonstandardized pulses to be significantly less than for standardized pulses. Since they do not observe this, it must indicate either that the pulse-height distribution is more peaked than they obtained or that they are losing many pulses on standardization or both.

Integrated Charge Distributions—the Effect of Storage Technique

For this measurement the standardized output from the FW130 was used into the amplifier which had either

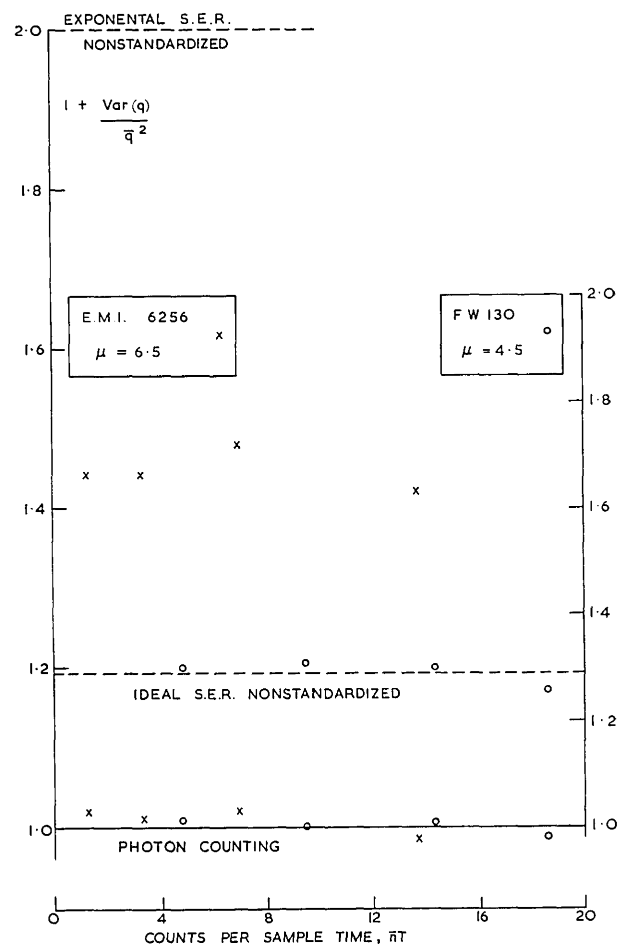

Fig. 6. The dependence of $1 + \mathrm{Var}\,(q)/q^{-2}$ on output standardization for both the FW130 and the 6256 with different numbers of counts per sample time. The high and low sets of points for each tube are for nonstandardized and standardized pulses, respectively. The ordinate scales are such that the ideal performance lies at the same reference level.

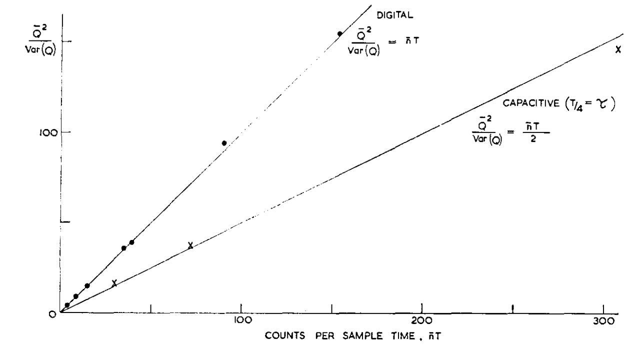

Fig. 7. The dependence of the relative variance of the integrated-charge distributions for a standardized output on the number of counts per sample time. The effect of the different storage techniques is shown. The continuous line is the theoretical prediction; the points are experimental.

a long internal time constant (capacitive storage) or an external reset (digital storage). The integrated-charge distributions were measured using as previously sampled voltage analysis. Since it was necessary to determine the signal in a time T the internal time constant was set at $\tau = T/4$, giving 2% distortion to the result, and samples were taken at intervals of T. As explained above, if less distortion had been required a smaller internal time constant could have been used at the expense of worse statistical accuracy. The results of a series of these measurements with different numbers of counts per sample time for capacitive and digital storage are in Fig. 7. It can be seen that the use of a capacitive store under these conditions does require double the integration time to achieve the desired result.

Conclusions

It has been shown experimentally that the relative variances of the integrated charge distributions are as predicted by the theory. Thus if we consider current measurements with the FW130 into a capacitive store, this will require a total length of experiment a factor of 2.6 longer than that required for photon counting into a digital store. This is the best performance theoretically possible with the given first-stage gain of 4.5. For the 6256 this factor rises slightly to become 2.9 with a first-stage gain of 6.5. With this first-stage gain the minimum theoretical value would be 2.4. The increase over this value reflects the nonzero b for this tube ($b = 0.23$).

The major contribution to the difference in experiment times, therefore, comes from the contrast between capacitive and digital stores (here a factor of 2). In the extreme case of a very poor multiplier having an exponential $P(q)$ the effect of standardization would also contribute a factor of 2. For each of the two tubes considered here, however, this latter contribution was much smaller. The additional integrating time needed in these cases may not seem significant where high light fluxes entail only short experiment times; for low light levels, however, the same factor can become important.[4]

Appendix

Digital Storage

Consider the total charge Q arriving from a train of infinitely short photomultiplier output pulses integrated for a time T. The probability of Q is given by

$$P(Q) = \sum_n P(Q|n)p(n,T), \tag{A1}$$

i.e., it is made up of the product of the probability of n pulses having a total charge Q with the probability of n pulses arriving in time T summed over all values of n.

The probability of n pulses having a total charge Q is the n-fold convolution of the individual charge probability distribution $P_q(q)$, i.e.,

$$P(Q|n) = P_q(q_1)*P_q(q_2)*P_q(q_3)\cdots \tag{A2}$$

Hence the moment generating function for $P(Q)$ is

$$\langle e^{-sQ}\rangle = \int e^{-sQ}P(Q)dQ \tag{A3}$$

$$= \sum_n p(n,T)\tilde{P}^n(s), \tag{A4}$$

where $\tilde{P}(s)$ is the Laplace transform of $P_q(q)$.

For broadband thermal, or coherent light, substitution for $p(n,T)$ gives

$$\langle e^{-sQ}\rangle = \sum_n \frac{(\bar{n}T)^n}{n!}e^{-(\bar{n}T)}\tilde{P}^n(s), \tag{A5}$$

where $\bar{n}$ is the mean number of counts per second. Since

$$\langle Q^m\rangle = (-1)^m \frac{d^m}{ds^m}\langle e^{-sQ}\rangle\Big|_{s=0} \tag{A6}$$

we have

$$\langle Q^2\rangle = \sum_n \frac{(\bar{n}T)^n e^{-(\bar{n}T)}}{n!}\left\{n(n-1)\tilde{P}^{n-2}(s)\left[\frac{d}{ds}P(s)\right]^2 + n\tilde{P}^{n-1}(s)(d^2/ds^2)P(s)\right\}\Bigg|_{s=0} \tag{A7}$$

$$= \sum_n \frac{(\bar{n}T)^n}{n!}e^{-(\bar{n}T)}\{n(n-1)\langle q^2\rangle + n\langle q^2\rangle\} \tag{A8}$$

$$= \bar{n}^2T^2\langle q\rangle^2 + \bar{n}T\langle q\rangle^2, \tag{A9}$$

Similarly,

$$\langle Q\rangle = \bar{n}\langle q\rangle T, \tag{A10}$$

and so

$$\mathrm{Var}(Q)/\bar{Q}^2 = (1/\bar{n}T)[1 + \mathrm{Var}(q)/\bar{q}^2]. \tag{A11}$$

Capacitive Storage

If we consider capacitive storage (time constant τ) such that the charge developed decays away then the charge Q_i measured at a time t_i is given by

$$Q_i = Y_i q_i \exp(-t_i/\tau), \tag{A12}$$

where Y_i equals 1 if a pulse were present or 0 if a pulse were absent during a sample at time t_i, of duration Δt sufficiently small for not more than one pulse to occur during Δt, q_i is the charge of a pulse arriving in this sample time.

The probability of obtaining no pulses during this sample is

$$P_i(0) = 1 - r\Delta t, \tag{A13}$$

where r is the mean count rate, and the probability of obtaining charge Q_i is given by

$$P_i(Q_i) = r\Delta t \exp(t_i/\tau)P[Q_i \exp(t_i/\tau)]. \tag{A14}$$

The factor $\exp(t_i/\tau)$ is required for correct normalization. Since the probability of obtaining a charge Q is, as before, the convolution of the individual charge probabilities, the moment generating function is

$$\langle e^{-sQ}\rangle = \bar{P}_1(s) \times \bar{P}_2(s) \times \ldots \tag{A15}$$

where

$$P_i(s) = \int \exp -(sQ_i)P_i(Q_i)dQ_i \tag{A16}$$

$$= 1 - r\Delta t + r\Delta t\int \exp\,(-sQ_i + t_i/\tau)P[Q_i \exp(t_i/\tau)]dQ_i. \tag{A17}$$

Putting

$$Q_i' = Q_i \exp\,(t_i/\tau) \tag{A18}$$

and substituting in Eq. (A17) we have, taking the limit as $\Delta t \rightarrow 0$,

$$\langle e^{-sQ}\rangle = \exp\left(-r\int_0^{+\infty}\left\{1 - \int \exp[-sQ\exp(-t/\tau)] \times P(Q)dQ\right\}dt\right) \tag{A19}$$

$$= \exp[-rt(s\langle q\rangle - (s^2/2.2!)\langle q^2\rangle + (s^3/3.3!)\langle q^3\rangle \ldots)]. \tag{A20}$$

Thus, using Eq. (A6),

$$\langle Q^2\rangle = \frac{r\tau}{2}\langle q^2\rangle + r^2\tau^2\langle q\rangle^2 \tag{A21}$$

and

$$\langle Q\rangle^2 = r^2\tau^2\langle q\rangle^2. \tag{A22}$$

Thus

$$\mathrm{Var}(Q)/\bar{Q}^2 = 1/2r\tau[1 + \mathrm{Var}(q)/\bar{q}^2]. \tag{A23}$$

A derivation considering only standardized pulses is given in Ref. 18, where the theory of the ratemeter is discussed.

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